**Propagation of Errors**

Let S be a square with side of length L. If we know the exact value of L, we could compute the exact area of S: area = L2. In the real world, we wouldn’t know the exact value of L. We would make multiple measurements of the length of the side and use the average, , to compute the area. But, measurements come with errors, so the sample mean will have an error. The error in  will produce an error in our calculated area. How does the error in affect the accuracy of our computed area?

**The general question**: If f(x) is a function of x, how does an error in the value of x propagate to produce the error in the value of f(x)?

**Standard Errors**

Suppose we make n measurements of the length, L, of the square. We would get n somewhat different values. We might expect that the average of our measurements, , would be close to L (the larger the number of measurements, the closer we would expect it would be). How can we quantify the expected error in? Use the standard deviation of  over all sets of n measurements. We can estimate this as , where s = standard deviation of our sample of n measurements. The quantity is called the **standard error** of our estimate: SE = .

**Example:** We make 25 measurements of L. The sample mean is = 3.41 cm and

s = 0.20 cm. Then, SE =  and we would report our estimate of L as

L = 3.41 ± .04 cm.

(Note that this is a confidence interval for L. It has the form ± t24, where t24 = 1. The value of t corresponds to a 67% CI.)

We would report the area of the square as

Area = 11.63 ± SE cm2

Where SE is the standard error in 2.

We know sum basic properties of variance:

Var(X+a) = Var(X)

Var(aX) = a2Var(X)

Unfortunatly, there is no formula for Var(X2). The good news is that there is a nice approximate formula.

Recall, from Calculus 1, any differentiable function f(x) can be approximated near a point (a,f(a)) using the tangent line at that point.

f(x) f(a) + f ’(a)(x-a) for x near a.

Let f(x) = x2. Let L be the actual side length of the square. Then is close to L, so

f() f(L) + f ’(L)( -L).

So, Var(f()) Var(f(L) + f ’(L)( -L))

= Var(f ’(L)( -L)))

=(f ’ (L))2\*Var( -L)

= (f ’ (L))2\*Var()

and

sd(f())  sqrt(Var(f()) = sqrt((f ’ (L))2\*Var())

= | f ’ (L) |\*sd()

= | f ’ (L) |\*SE()

So, for any differentiable function f,

SE(f())| f ’ (L) |\*SE().

The problem with this formula is that we don’t know L, so we cannot compute the right side. That is not a BIG problem, We know that is close to L, so we still get an approximation by

SE(f())| f ’ () |\*SE().

**General Formula**

**SE(f(x)) |f ‘(x)| \* SE(x)**

If f(x) = x2 , then f’(x) = 2x. So, SE(area) |2|\*SE() = 2\*3.41\*0.04 = 0.27

So, our report of the area of the square is

**Area = 11.63 ± 0.27**

**Example:** Find the volume of a cube of edge length x. 10 measurements of the edge produce a sample mean = 3.11 in. and a sample standard deviation s = .13 in. How should the volume of the cube be reported?

SE() = s/sqrt(10) = .04

Volume() = 3.113 ± SE(3) = 30.08 ± SE(3).

Since volume(x) = f(x) = x3 and f ‘(x) = 3x2,

SE(3)  |32|\*SE() = 29.02\*.04 = 1.16. So, the final report is

**Volume = 30.08 ± 1.16 in3**

**Example:** An angle has size , measured in radians. The exact value of isn’t known, but a sample of size n = 20 produces a sample mean = 1.21. and a sd s = 0.09. How should the value of sin() be reports?

In this case f(x) = sin(x) and f ‘(x) = cos(x), so SE(sin()) |cos()|\*SE().

SE() = 0.9 /sqrt(20) = 0.02, cos(1.21) = .353, so SE(sin()) .01

Since sin(1.21 ) = .94, the report would be

**sin() = 0.94 ± 0.01**

**Example.** An angle = 1.52 ± .01 (in radians), What is the value of tan()?

tan() = tan(1.52) ± SE(tan()) = 19.67 ± SE(tan()).

Since the dervative of tan is sec2,

SE(tan()) |sec2(1.52)|\*SE() = 387.89\*0.01 = 3.88.

So, **tan() = 19.67 ± 3.88**

**The Two Variable Case**

What if we are combining two quantities? For example, what is we have estimates of the length and width of a rectangle and want to use them to get the area of the rectangle?

Suppose is an estimate for x (with a standard error SE() )and is an estimate for y (with a standard error SE()) and and are independent of each other. Then,

SE(f(,))  (Delta Method)

This formula involves partial derivative of the function f(x,y). It comes from the fact that the graph of a function of two variables can be approximated at a point by the tangent plane to the point.

**Example** Suppose 3.41± .04 is the estimate of the length of a rectangle and 2.34 ± .02 is the estimate of its lwidth. How do we report the area?

Area = f(x,y) = xy. fx(x,y) = y, fy(x,y) = x

Area = f(,) ± SE(f(,)) = 3.41\*2.34 ± SE(f(,)) = 7.98 ± SE(f(,))

SE(f(,)) 

**=**

= .12

**The final result: Area = 7.98 ± 0.12**

**Example**: The length of a rectangle is given by = 3.05 ± .02 and the width of the rectangle is given by = 5.45 ± .11. How should perimeter of the rectangle be reported?

Perimeter = f(x,y) = 2x + 2y, fx (x,y) = 2, fy(x,y) = 2

Perimeter = f(,) ± SE(f(,)) = 2\*3.05+2\*5.45 ± SE(f(,)) = 17.00 ± SE(f(,))

SE(f(,)) 

=

= .22

**The final result: Perimeter = 17.00 ± .22.**

**Exercises 18**

1. The angle  has size .32 ± .02 (in radians) What is the cos()?
2. The radius of a circle is reported as r = 3.45 ± .06. in. What is the area of the circle?
3. The radius of a sphere is given as r = 4.23 ± .10 in. What is the volume of the sphere